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ALGEBRA HOMOMORPHISMS FROM COSINE CONVOLUTION ALGEBRAS*

BY

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ABSTRACT

In this paper we deal with the weighted Banach algebra $L^1_\omega(\mathbb{R}^+, *_c)$, where $*_c$ is the cosine convolution product. We describe its character space and its multiplier algebra. Our main results concern bounded algebra homomorphisms from $L^1_\omega(\mathbb{R}^+, *_c)$. We give a variant of Kisyński's theorem for such homomorphisms and characterize them in terms of integrated cosine functions. A generalized form of the Sova-Da Prato-Giusti theorem about generation of cosine functions is also given.

Introduction

Let $\mathbb{R}, \mathbb{R}^+$ and $\mathbb{C}$ be the sets of real, non-negative real and complex numbers respectively, and let μ be a non-negative Borel measure on $\mathbb{R}^+$. As usual, the Banach space $(L^1_\mu(\mathbb{R}^+), \| \ \|_\mu)$ is the set (of classes) of Lebesgue-measurable functions, $f : \mathbb{R}^+ \to \mathbb{C}$, such that

$$\|f\|_\mu := \int_0^\infty |f(t)| d\mu(t) < \infty.$$

A bound for μ is an element $\kappa \in \{-\infty\} \cup \mathbb{R}$ such that $(\epsilon_{-\lambda})_{\lambda > \kappa} \subset L^1_\mu(\mathbb{R}^+)$, where $\epsilon_{-\lambda}(t) := e^{-\lambda t}$ for $t \in \mathbb{R}^+$.

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For two Banach spaces X and Y, we denote by $\mathcal{B}(X,Y)$ the Banach space of bounded linear operators from X to Y. Put $\mathcal{B}(X) := \mathcal{B}(X,X)$. Recently, W. Chojnacki has established the following result.

Theorem 0.1 ([5], Theorem 1.2): *Let μ be a non-negative Borel measure on $\mathbb{R}^+$ with bound κ, X a Banach space, and $r : (\kappa,\infty) \to X$ a function. Then the following conditions are equivalent:*

(i) *There exists $T \in \mathcal{B}(L^1_\mu(\mathbb{R}^+), X)$ such that $r(\lambda) = T(\epsilon_{-\lambda})$ for $\lambda > \kappa$.*

(ii) *The function r belongs to $C^{(\infty)}((\kappa,\infty),X)$ and satisfies*

$$\|r\|_{W,\mu,\kappa} := \sup\left\{ \frac{\|r^{(n)}(\lambda)\|}{\|t^n e^{-\lambda t}\|_\mu} : n \in \mathbb{N}\cup\{0\}, \lambda \in (\kappa,\infty)\right\} < \infty.$$

Moreover, if there exists T as in (i), *then T is unique and $\|T\| = \|r\|_{W,\mu,\kappa}$.*

Particularly interesting applications of the above theorem arise when one considers spaces $L^1_\mu(\mathbb{R}^+)$ endowed with an algebraic structure. Let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be a weight function, i.e., a continuous function such that $\omega(s+t) \le C\omega(s)\omega(t)$ for $s,t \ge 0$ for a constant $0 < C < \infty$. Then there exists $\kappa \in \mathbb{R}$ and $M > 0$ such that $\omega(t) \le Me^{\kappa t}$ for $t > 0$. Moreover, the Lebesgue space $(L^1_\omega(\mathbb{R}^+), \| \quad \|_\omega)$ is a Banach algebra with respect to the convolution $*$ defined by

$$f * g(t) := \int_0^t f(t-s)g(s)ds, \quad f,g \in L^1_\omega(\mathbb{R}^+),$$

and the norm given by

$$\|f\|_\omega := \int_0^\infty |f(t)|\omega(t)dt < \infty, \quad f \in L^1_\omega(\mathbb{R}^+).$$

We denote this algebra by $L^1_\omega(\mathbb{R}^+,*)$ (in particular, $L^1(\mathbb{R}^+,*)$ for $\omega(t)=1$).

Let $\mathcal{A}$ be a Banach algebra. A continuous function $r : (\kappa,\infty) \to \mathcal{A}$ is a pseudo-resolvent if the equation

$$r(\lambda) - r(\mu) = (\mu - \lambda) r(\lambda) r(\mu)$$

holds for $\lambda, \mu > \kappa$. If $(\epsilon_{-\lambda})_{\lambda>\kappa} \subset L^1_\omega(\mathbb{R}^+)$, then $(\epsilon_{-\lambda})_{\lambda>\kappa}$ is a pseudo-resolvent in $L^1_\omega(\mathbb{R}^+,*)$.

J. Kisyński was the first one to notice a connection between algebra homomorphisms (in particular representations) and pseudo-resolvents, see [3] and [9]. The following theorem characterizes certain classes of pseudo-resolvents and shows that $(\epsilon_{-\lambda})_{\lambda>\kappa}$ is a canonical pseudo-resolvent for these classes.

THEOREM 0.2 ([5] Theorem 5.1): *Let $\mathcal{A}$ be a Banach algebra, ω a weight function on $\mathbb{R}^+$ with bound κ, and $r : (\kappa, \infty) \to \mathcal{A}$, $\lambda \mapsto r(\lambda)$ a pseudo-resolvent. Then the following conditions are equivalent:*

(i) *There exists a bounded algebra homomorphism $T \in \mathcal{B}(L^1_\omega(\mathbb{R}^+), \mathcal{A})$ such that $r(\lambda) = T(\epsilon_{-\lambda})$ for $\lambda > \kappa$.*

(ii) *The function r satisfies the Hille–Yosida condition*

$$\|r\|_{W,\omega,\kappa} = \sup\left\{\frac{n!\ \|r^{n+1}(\lambda)\|}{\|t^n e^{-\lambda t}\|_\omega} : n \in \mathbb{N} \cup \{0\}, \lambda \in (\kappa, \infty)\right\} < \infty.$$

Moreover, if there exists T as in (i)*, then T is unique and $\|T\| = \|r\|_{W,\omega,\kappa}$.*

Kisyński's point of view allows generalizations of the Trotter–Kato theorem and has interesting applications in the generation of one-parameter and integrated semigroups. In fact, a first version of Theorem 0.1 was derived directly from the Hille–Yosida theorem in [3].

In this paper we consider the cosine convolution product $*_c$ in the Banach space $L^1_\omega(\mathbb{R}^+)$ for certain weight functions ω, see Section 1. We describe the character space of the Banach algebra $L^1_\omega(\mathbb{R}^+, *_c)$ and show that its multiplier algebra $Mul(L^1_\omega(\mathbb{R}^+, *_c))$ is isomorphic to $M_\omega(\mathbb{R}^+, *_c)$ (Theorem 2.6). Here $M_\omega(\mathbb{R}^+)$ is the space of all Borel measures on $\mathbb{R}^+$ such that

$$\|\mu\|_\omega := \int_{\mathbb{R}^+} \omega(t) d|\mu|(t) < \infty,$$

where $|\mu|$ denotes the total variation of μ.

In the third section we prove a variant of Kisyński's theorem for algebra homomorphisms from $L^1_\omega(\mathbb{R}^+, *_c)$ (Theorem 3.1). Following similar ideas to those of the case $L^1_\omega(\mathbb{R}^+, *)$ in [12], we consider in Section 4 integrated cosine functions, fractional Banach algebras $\mathcal{T}_+^{(\alpha)}(\omega_\alpha, *_c)$ (which are contained in $L^1_\omega(\mathbb{R}^+, *_c)$) and uniformly bounded limits of fractional homomorphisms, in order to characterize algebra homomorphisms from $L^1_\omega(\mathbb{R}^+, *_c)$ into a Banach algebra $\mathcal{A}$ (Theorem 4.8).

In the last section we prove a generalization of the generation theorem for cosine functions on a Banach space (Theorem 5.3). The generation theorem for cosine functions has been established by M. Sova in [17] and G. Da Prato and E. Giusti in [7].

Our approach is closer to that taken in a paper of A. Bobrowski in [2]. Bobrowski's paper exploits the subalgebra $L^1_{e,\Omega}(\mathbb{R})$ of even functions in the Banach algebra $L^1_\Omega(\mathbb{R})$, where $\Omega:\mathbb{R}\to\mathbb{R}^+$ is a symmetric weight function, the norm is defined by

$$\|F\|_\Omega := \int_{-\infty}^{\infty} |F(t)|\Omega(t)dt < \infty, \quad F\in L^1_\Omega(\mathbb{R}),$$

and the convolution product $*$ given by

$$F*G(t) := \int_{-\infty}^{\infty} F(s-t)G(s)ds, \quad F,G\in L^1_\Omega(\mathbb{R}).$$

Although $L^1_{e,\Omega}(\mathbb{R})$ is isomorphic to $L^1_\omega(\mathbb{R}^+,*_c)$ with ω the restriction of Ω to $\mathbb{R}^+$, we prefer to make use of the former rather than the latter. We work directly with the structure of $\mathbb{R}^+$ without considering $\mathbb{R}^+$ inside of $\mathbb{R}$. Both papers share a common spirit and some similar results are obtained after different starting points (compare Theorem 3.1 and [2, Proposition 3.1]).

Notation: For $z\in\mathbb{C}$, $\Re z$ denotes the real part of z and $\Im z$ denotes its imaginary part. Let X be a Banach space and T a linear (bounded or unbounded) operator on X. Let $\rho(T)$ denote the resolvent set of T and $(\lambda-T)^{-1}$ the resolvent operator for $\lambda\in\rho(T)$.

1. The Banach algebra $L^1_\omega(\mathbb{R}^+,*_c)$

In this section we consider in detail the Banach algebra $L^1_\omega(\mathbb{R}^+,*_c)$: we show that it has bounded approximate identities and its character space may be identified with a half strip of complex numbers.

Definition 1.1: A continuous map $0\neq\omega:\mathbb{R}^+\to\mathbb{R}^+$ is an **extendible weight function** if it satisfies

$$\omega(t+s)\le C\omega(t)\omega(s), \qquad \omega(t-s)\le C\omega(t)\omega(s), \qquad 0\le s\le t,$$

for some constant $C>0$. The infimum of such constants is called the **growth constant** of ω.

Note that $\omega:\mathbb{R}^+\to\mathbb{R}$ is an extendible weight function if and only if the function $\Omega:\mathbb{R}\to\mathbb{R}^+$ defined by

$$\Omega(t) := \begin{cases} \omega(t), & t\ge 0, \\ \omega(-t), & t\le 0, \end{cases} \tag{1.1}$$

is a weight function on $\mathbb{R}$. Examples of extendible weight functions are $\omega(t)=e^{\kappa t}$; $\omega(t)=(1+t)^\gamma$ and $\omega(t)=e^{\kappa t^\gamma}$ with $\kappa,\gamma,t\ge 0$.

The proof of the following lemma is straightforward.

Lemma 1.2: *Let ω be an extendible weight function.*

(i) *Then $\omega(t) \neq 0$ holds for any $t \geq 0$.*

(ii) *There exists $M > 0$ such that $\omega(t) \geq M$ for any $t \geq 0$.*

Define a convolution product $\circ$ in the Banach space $L^1_\omega(\mathbb{R}^+)$ by

$$f \circ g(t) := \int_t^\infty f(s-t)g(s)ds,$$

for $t \geq 0$ and $f, g \in L^1_\omega(\mathbb{R}^+)$. It is easy to check that $f \circ g \in L^1_\omega(\mathbb{R}^+)$, the product $\circ$ is non-commutative (see Example 1) and $\|f \circ g\|_\omega \leq \|f\|_\omega \|g\|_\omega$. Products $*$ and $\circ$ are dual in the sense that

$$\int_0^\infty h(t)(f * g)(t)dt = \int_0^\infty f(t)(g \circ h)(t)dt, \tag{1.2}$$

for $f, g \in L^1_\omega(\mathbb{R}^+)$ and $h \in L^\infty_\omega(\mathbb{R}^+)$, where the Lebesgue space $L^\infty_\omega(\mathbb{R}^+)$ is the dual Banach space of $L^1_\omega(\mathbb{R}^+)$ given by

$$L^\infty_\omega(\mathbb{R}^+) := \{f : \mathbb{R}^+ \to \mathbb{C} \,|\, \omega(t)^{-1}|f(t)| \text{ is bounded for almost every } t \geq 0\}.$$

Then the **cosine convolution product** $*_c$ is defined by

$$f *_c g := \frac{1}{2}\left(f * g + f \circ g + g \circ f\right), \quad f, g \in L^1_\omega(\mathbb{R}^+),$$

see for example [16].

Proposition 1.3: *Let ω be an extendible weight function on $\mathbb{R}^+$ with growth constant C and $f, g \in L^1_\omega(\mathbb{R}^+)$. Then:*

(i) $\|\, |f| \circ |g| + |g| \circ |f| \,\|_\omega \leq C\|f\|_\omega \|g\|_\omega$.

(ii) $\|f *_c g\|_\omega \leq C\|f\|_\omega \|g\|_\omega$.

Proof. First we use definitions of the cosine convolution product and extendible weight functions and then we apply the Fubini theorem. ∎

Let $L^1_\omega(\mathbb{R}^+, *_c)$ denote the Banach algebra which results from providing $L^1_\omega(\mathbb{R}^+)$ with the cosine convolution product $*_c$. It is straightforward that

$$f *_c g(t) = \frac{1}{2}(F * G)(t), \quad t \geq 0, \tag{1.3}$$

where $F, G : \mathbb{R} \to \mathbb{C}$ are defined by $F(t) := f(|t|)$, $G(t) := g(|t|)$ for $t \in \mathbb{R}$, $F, G \in L^1_\Omega(\mathbb{R})$, and Ω is given by (1.1).

The cosine convolution product $*_c$ has been considered by several authors, see for example [16] and [18]; for $\omega(t)=e^{\kappa t}$ with $\kappa\geq 0$ see [13]. However, the algebraic structure of $L^1_\omega(\mathbb{R}^+,*_c)$ has not been studied in detail yet. Firstly we check cosine convolution products of some known functions.

Example 1: Let $\omega:\mathbb{R}^+\to\mathbb{R}$ be an extendible weight function with bound $\kappa\geq 0$. Then the exponential function $\epsilon_{-\lambda}$ belongs to $L^1_\omega(\mathbb{R}^+)$ for any $\Re\lambda>\kappa$, and satisfies

$$\epsilon_{-\lambda}\circ\epsilon_{-\nu}=\frac{1}{\lambda+\nu}\epsilon_{-\nu},\quad \epsilon_{-\lambda}*\epsilon_{-\nu}=\frac{1}{\nu-\lambda}\left(\epsilon_{-\lambda}-\epsilon_{-\nu}\right),$$

and

$$\epsilon_{-\lambda}*_c\epsilon_{-\nu}=\frac{1}{\lambda^2-\nu^2}\left(\lambda\epsilon_{-\nu}-\nu\epsilon_{-\lambda}\right),\tag{1.4}$$

whenever $\Re\lambda,\Re\nu>\kappa$. Moreover, the linear space spanned by the set $\{\epsilon_{-\nu}\colon\nu>\kappa\}$ is dense in $L^1_\omega(\mathbb{R}^+)$, i.e., the set $\{\epsilon_{-\nu}\colon\nu>\kappa\}$ is *total* in $L^1_\omega(\mathbb{R}^+)$ ([5, Proposition 2.2]).

Example 2: Set $\omega(t)=(1+t)^\gamma$ with $0\leq\gamma<1$. The **Poisson semigroup** $(P^z)_{\Re z>0}$ in $L^1_{(1+|t|)^\gamma}(\mathbb{R})$ is given by

$$P^z(t):=\frac{1}{\pi}\frac{z}{z^2+t^2},\quad t\in\mathbb{R},\ \Re z>0.$$

Put $p^z(t):=2P^z(t)$ for $t\geq 0$. Then $(p^z)_{\Re z>0}\subset L^1_{(1+t)^\gamma}(\mathbb{R}^+,*_c)$ and satisfies the semigroup law, $p^z*_c p^{z'}=p^{z+z'}$. This is due to the fact that

$$p^z*_c p^{z'}=2(P^z*P^{z'})=2P^{z+z'}=p^{z+z'},$$

where we have used equality (1.3). The following estimate is readily seen:

$$\|p^s\|_{(1+t)^\gamma}\leq C(1+s^\gamma),\quad s>0.$$

Example 3: Let ω be an extendible weight function such that $\omega(t)\leq Me^{\kappa t}$ for $t\geq 0$ and $\kappa>0$. Let Ω be as in (1.1). The Gaussian semigroup $(G^z)_{\Re z>0}$ in $L^1_\Omega(\mathbb{R})$ is defined by

$$G^z(t):=1\sqrt{4\pi z}e^{-t^2/4z},\quad t\in\mathbb{R},\ \Re z>0.$$

Define $g^z(t):=2G^z(t)$ for $t\geq 0$. Then $(g^z)_{\Re z>0}\subset L^1_\omega(\mathbb{R}^+,*_c)$ and the following semigroup law holds,

$$g^z*_c g^{z'}=g^{z+z'},\quad \Re z,\Re z'>0.$$

Moreover,

$$\|g^z\|_\omega \leq M\big(|z|/(\Re z)\big)^{1/2} e^{\kappa^2|z|^2/\Re z}, \quad \Re z > 0.$$

PROPOSITION 1.4: *Let ω be an extendible weight function on $\mathbb{R}^+$. The Banach algebra $L^1_\omega(\mathbb{R}^+, *_c)$ has a bounded approximate unit.*

Proof. By Example 3, $(g^z)_{\Re z>0} \subset L^1_\omega(\mathbb{R}^+, *_c)$ and $\|g^s\|_\omega \leq C'$ for $s \in (0,1)$. We have to check that $g^s *_c f \to f$ for $f \in L^1_\omega(\mathbb{R}^+, *_c)$ when $s \to 0^+$. Let Ω be given by equation (1.1) and put $F(t) := f(|t|)$ for $t \in \mathbb{R}$. Then

$$\|g^s *_c f - f\|_\omega = \int_0^\infty |G^s * F(t) - F(t)|\omega(t)dt = \frac{1}{2}\|G^s * F - F\|_\Omega \to 0,$$

where we have used the fact that $(G^s)_{0<s<1}$ is a bounded approximate unit on $L^1_\Omega(\mathbb{R})$. ∎

For ω an extendible weight function, let

$$\sigma_\omega := \lim_{t\to\infty} \log\omega(t)/t = \inf_{t>1} \log\omega(t)/t.$$

By the assertion (ii) of Lemma 1.2, the inequality $\sigma_\omega \geq 0$ holds. It is well-known that the Banach algebra $L^1_\omega(\mathbb{R}^+, *)$ is semisimple, its character space $\triangle(L^1_\omega(\mathbb{R}^+, *))$ is homeomorphic to the half plane $\overline{\Pi}_{-\sigma_\omega}$ defined as

$$\overline{\Pi}_{-\sigma_\omega} := \{z \in \mathbb{C} \colon \Re z \geq -\sigma_\omega\},$$

and the Gelfand transform is the Laplace transform $\mathcal{L} : L^1_\omega(\mathbb{R}^+, *_c) \to C_0(\overline{\Pi}_{-\sigma_\omega})$ given by

$$\mathcal{L}(f)(z) = \int_0^\infty f(t)e^{-zt}dt, \quad z \in \overline{\Pi}_{-\sigma_\omega};$$

see these results, for example, in [6, Theorem 4.7.27].

Now we come back to the Banach algebra $L^1_\Omega(\mathbb{R})$ defined in the introduction. It is semisimple, its character space $\triangle(L^1_\Omega(\mathbb{R}))$ is homeomorphic to the strip

$$\overline{\Pi}_{-\sigma_\omega,\sigma_\omega} := \{z \in \mathbb{C} \colon -\sigma_\omega \leq \Re z \leq \sigma_\omega\},$$

and the Gelfand transform is the bilateral Laplace transform $\mathcal{L} : L^1_\Omega(\mathbb{R}) \to C_0(\overline{\Pi}_{-\sigma_\omega,\sigma_\omega})$ defined as

$$\mathcal{L}(F)(z) = \int_{-\infty}^\infty F(t)e^{-zt}dt, \quad z \in \overline{\Pi}_{-\sigma_\omega,\sigma_\omega},$$

see [6, Theorem 4.7.33].

THEOREM 1.5: *Let ω be an extendible weight function. Then $L^1_\omega(\mathbb{R}^+, *_c)$ is a semisimple Banach algebra and its character space $\triangle(L^1_\omega(\mathbb{R}^+, *_c))$ can be identified with the half strip $\overline{\Pi}^+_{-\sigma_\omega,\sigma_\omega}$ where*

$$\overline{\Pi}^+_{-\sigma_\omega,\sigma_\omega} := \{z \in \mathbb{C}\colon -\sigma_\omega \leq \Re z \leq \sigma_\omega; \Im z \geq 0\}.$$

The Gelfand transform is given by the cosine transform

$$\mathcal{C} : L^1_\omega(\mathbb{R}^+, *_c) \to C_0(\overline{\Pi}^+_{-\sigma_\omega,\sigma_\omega}),$$

thus

$$\mathcal{C}(f)(z) := \int_0^\infty f(t)\cosh(zt)dt, \quad z \in \overline{\Pi}^+_{-\sigma_\omega,\sigma_\omega}.$$

Proof. If $\kappa > \sigma_\omega$, then $\omega(t) \leq Ce^{\kappa t}$ for $t \geq 0$, and $(\epsilon_{-\lambda})_{\Re\lambda>\kappa} \subset L^1_\omega(\mathbb{R}^+)$. Take $\chi \in \triangle(L^1_\omega(\mathbb{R}^+, *))$ and define

$$g(\lambda) := \chi(\epsilon_{-\lambda}), \quad \Re\lambda > \kappa.$$

Note that $g \neq 0$: if $g = 0$ then we use the fact that the set $(\epsilon_{-\lambda})_{\Re\lambda>\kappa}$ is total on $L^1_\omega(\mathbb{R}^+)$ to conclude that $\chi = 0$, which is a contradiction. Using Example 1, we have

$$g(\lambda)g(\nu) = \frac{1}{\lambda^2 - \nu^2}(\lambda g(\nu) - \nu g(\lambda)), \quad \Re\lambda, \Re\nu > \kappa. \tag{1.5}$$

Since $g \neq 0$, it follows that $g(\kappa + 1) \neq 0$ (otherwise, using the equality (1.5) we conclude that $g = 0$). Taking $\nu = \kappa + 1$ and $z := \frac{\kappa+1}{g(\kappa+1)} - (\kappa + 1)^2$, we obtain from the equality (1.5) that

$$g(\lambda) = \lambda/(\lambda^2 + z), \quad \Re\lambda > \kappa.$$

Take now $z = -u^2$ with $u \in \mathbb{C}$. Since the character χ is continuous, we see that

$$|g(\lambda)| = \left|\frac{\lambda}{\lambda^2 - u^2}\right| = \frac{|\lambda|}{|\lambda - u|}\frac{1}{|\lambda + u|} \leq \|\chi\|\,\|\epsilon_{-\lambda}\|_\omega \leq \frac{\|\chi\|}{\Re\lambda - \kappa},$$

for $\Re\lambda > \kappa$. Now suppose that $|\Re u| > \kappa$. Then we take $\lambda = u$ or $\lambda = -u$ to obtain a contradiction with the above inequalities. We conclude that $-\kappa \leq \Re u \leq \kappa$ and $-\sigma_\omega \leq \Re u \leq \sigma_\omega$. Moreover, we have

$$\chi(\epsilon_{-\lambda}) = \frac{\lambda}{\lambda^2 - u^2} = \int_0^\infty \epsilon_{-\lambda}(t)\cosh(ut)dt,$$

for $\Re\lambda > \sigma_\omega$. Since the set $(\epsilon_{-\lambda})_{\Re\lambda>\kappa}$ is total on $L^1_\omega(\mathbb{R}^+)$, we obtain

$$\chi(f) = \int_0^\infty f(t)\cosh(ut)dt = \mathcal{C}(f)(u), \quad f \in L^1_\omega(\mathbb{R}^+, *_c).$$

Note that $\mathcal{C}(f)(u) = \mathcal{C}(f)(-u)$ for $u \in \overline{\Pi}_{-\sigma_\omega,\sigma_\omega}$. In conclusion, given $\chi \in \triangle(L^1_\omega(\mathbb{R}^+,*))$ there exists $u \in \overline{\Pi}^+_{-\sigma_\omega,\sigma_\omega}$ such that

$$\chi(f) = \mathcal{C}(f)(u), \quad f \in L^1_\omega(\mathbb{R}^+, *_c).$$

Conversely, it is readily seen that $f \mapsto \mathcal{C}(f)(u)$ is a character on $L^1_\omega(\mathbb{R}^+, *_c)$ with $u \in \overline{\Pi}^+_{-\sigma_\omega,\sigma_\omega}$, see similar ideas about cosine transforms in [13]. Thus the character space $\triangle(L^1_\omega(\mathbb{R}^+, *_c))$ can be identified with the set $\overline{\Pi}^+_{-\sigma_\omega,\sigma_\omega}$.

In order to show that $L^1_\omega(\mathbb{R}^+, *_c)$ is a semisimple Banach algebra, it is enough to check that $\mathcal{C} : L^1_\omega(\mathbb{R}^+, *_c) \to C_0(\overline{\Pi}^+_{-\sigma_\omega,\sigma_\omega})$ is injective, see for example [6, Corollary 2.3.26]. Take $f \in L^1_\omega(\mathbb{R}^+, *_c)$ such that $\mathcal{C}(f) = 0$. Then

$$0 = \mathcal{C}(f)(u) = \frac{1}{2}\int_{-\infty}^{\infty} F(t)e^{-ut}dt = \frac{1}{2}\mathcal{L}(F)(u), \quad u \in \overline{\Pi}_{-\sigma_\omega,\sigma_\omega},$$

where $F(t) := f(|t|)$ for $t \geq 0$. By the analytic continuation principle, we have $\mathcal{L}(F)(u) = 0$ for every $u \in \overline{\Pi}_{-\sigma_\omega,\sigma_\omega}$. Since $L^1_\Omega(\mathbb{R})$ is semisimple, it follows that $F = 0$ and $f = 0$. ∎

2. The Banach algebra $Mul(L^1_\omega(\mathbb{R}^+, *_c))$

For a commutative Banach algebra $\mathcal{A}$, let $Mul(\mathcal{A})$ denote the subspace of bounded linear operators on $\mathcal{A}$, $T : \mathcal{A} \to \mathcal{A}$, such that

$$T(ab) = aT(b), \quad a, b \in \mathcal{A}.$$

The space $Mul(\mathcal{A})$ is a Banach algebra with respect to the composition and the operator norm. It is usually called the **multiplier algebra** of $\mathcal{A}$.

In the case that ω and Ω are weight functions on $\mathbb{R}^+$ and $\mathbb{R}$ respectively, the multiplier algebras of $L^1_\omega(\mathbb{R}^+, *)$ and $L^1_\Omega(\mathbb{R})$ may be identified with the space of Borel measures on $\mathbb{R}^+$ and $\mathbb{R}$ of total variation,

$$Mul(L^1_\omega(\mathbb{R}^+, *)) \cong M_\omega(\mathbb{R}^+, *), \quad Mul(L^1_\Omega(\mathbb{R})) \cong M_\Omega(\mathbb{R}),$$

where $\cong$ means "isomorphic" as Banach algebras, see for example [4] and [6]. We remind that the convolution product $\mu * \nu$ of two measures $\mu, \nu \in M_\omega(\mathbb{R}^+)$ is defined by

$$(\mu * \nu)(A) := \int_{\mathbb{R}^+} \mu(A \ominus t)d\nu(t),$$

where $A \ominus t = \{s \in \mathbb{R}^+ \colon s = a - t \text{ for some } a \in A\}$ for a Borel set A on $\mathbb{R}^+$ and $\mu * \nu \in M_\omega(\mathbb{R}^+)$, see for example [4]. In this section we show that

$$Mul(L^1_\omega(\mathbb{R}^+, *_c)) \cong M_\omega(\mathbb{R}^+, *_c).$$

Let $C_{b,\omega}(\mathbb{R}^+)$ be the space of all continuous functions f on $\mathbb{R}^+$ such that there exists $M > 0$ with

$$|f(t)| \leq M\omega(t) \quad \text{ for all } t \geq 0,$$

and $C_{0,\omega}$ be the space of all continuous functions f on $\mathbb{R}^+$ for which

$$\lim_{t\to\infty} f(t)/\omega(t) = 0.$$

Endowed with the norm

$$\|f\|_{\infty,\omega} := \sup\{|f(t)|/\omega(t) \colon t \geq 0\},$$

$C_{b,\omega}(\mathbb{R}^+)$ and $C_{0,\omega}(\mathbb{R}^+)$ are Banach spaces. It is well-known that the topological dual space of $C_{0,\omega}(\mathbb{R}^+)$, $(C_{0,\omega}(\mathbb{R}^+))'$, is isomorphic to $M_\omega(\mathbb{R}^+)$. In fact, for every $T \in (C_{0,\omega}(\mathbb{R}^+))'$ there exists a unique $\mu \in M_\omega(\mathbb{R}^+)$ such that

$$T(f) = \int_0^\infty f(t) d\mu(t), \quad f \in C_{0,\omega}(\mathbb{R}^+),$$

see [4]. Let $\mu \in M_\omega(\mathbb{R}^+)$ and $f \in L^1_\omega(\mathbb{R}^+)$. Then $\mu * f \in L^1_\omega(\mathbb{R}^+)$ where

$$\mu * f(t) := \int_0^t f(t-s) d\mu(s), \quad t \in \mathbb{R}^+,$$

and $\|\mu * f\|_\omega \leq C\|\mu\|_\omega \|f\|_\omega$. We now consider other convolution products.

Definition 2.1: Let $\mu \in M_\omega(\mathbb{R}^+)$ and $f \in L^1_\omega(\mathbb{R}^+, *_c)$. We define $f \circ \mu, \mu \circ f$ and $\mu *_c f$ by

$$f \circ \mu(t) := \int_t^\infty f(t-s) d\mu(s), \quad \mu \circ f(t) := \int_0^\infty f(s+t) d\mu(s),$$

$$\mu *_c f(t) := \frac{1}{2}\left(\mu * f + \mu \circ f + f \circ \mu\right)(t),$$

for $t \geq 0$.

Note that $f *_c \delta_0 = f$ where δ_0 is the Dirac measure on 0 and $f \in L^1_\omega(\mathbb{R}^+, *_c)$. The following proposition is readily established.

PROPOSITION 2.2: *Let ω be an extendible weight function on $\mathbb{R}^+$ with growth constant C, $\mu \in M_\omega(\mathbb{R}^+)$ and $f, g \in L^1_\omega(\mathbb{R}^+)$. Then:*

(i) $f \circ \mu, \mu \circ f, \mu *_c f \in L^1_\omega(\mathbb{R}^+)$;

(ii) $\mu *_c (f *_c g) = f *_c (\mu *_c g)$;
(iii) $\| \mu \circ f + f \circ \mu\|_\omega \leq C\|\mu\|_\omega\|f\|_\omega$;
(iv) $\|\mu *_c f\|_\omega \leq C\|\mu\|_\omega\|f\|_\omega$.

The next lemma is needed in order to prove the main result of this section.

Lemma 2.3: *Let ω be an extendible weight function on $\mathbb{R}^+$, $h \in C_{0,\omega}(\mathbb{R}^+)$ and $f, g \in L^1_\omega(\mathbb{R}^+)$. Then:*
(i) $h * f, h \circ f, f \circ h, h *_c f \in C_{0,\omega}(\mathbb{R}^+)$;
(ii) $\int_0^\infty h(t)(f *_c g)(t)dt = \int_0^\infty f(t)(h *_c g)(t)dt$.

Proof. (i) It is clear that functions $h * f, h \circ f, f \circ h, h *_c f$ are continuous. Moreover, we use the property that $\omega(t-s) \leq C\omega(t)\omega(s)$ to prove that

$$|(h * f)(t)| \leq \int_0^t |h(t-s)|\,|f(s)|ds = C\omega(t)\int_0^t \frac{|h(t-s)|}{\omega(t-s)}\omega(s)|f(s)|ds,$$

for $t \geq 0$ and $h * f \in C_{0,\omega}(\mathbb{R}^+)$; in a similar way we prove that $h \circ f, f \circ h \in C_{0,\omega}(\mathbb{R}^+)$ and so $h *_c f \in C_{0,\omega}(\mathbb{R}^+)$.

By part (i) and Fubini's theorem, we obtain

$$\int_0^\infty h(t)(f \circ g)(t)dt = \int_0^\infty f(t)(h \circ g)(t)dt,$$
$$\int_0^\infty h(t)(g \circ f)(t)dt = \int_0^\infty f(t)(g * h)(t)dt.$$

With these two equalities and (1.2) we conclude the proof of (ii). ∎

Definition 2.4: Let A be a Borel set on $\mathbb{R}^+$ and $t \in \mathbb{R}^+$. We define the Borel set $t \oplus A$ as

$$t \oplus A := \{s \in \mathbb{R}^+ \colon s = t + a \ \text{ for some } a \in A\}.$$

Given $\mu, \nu \in M_\omega(\mathbb{R}^+)$, define $\mu \circ \nu, \mu *_c \nu \in M_\omega(\mathbb{R}^+)$ by

$$\mu \circ \nu(A) := \int_{\mathbb{R}^+} \nu(t \oplus A)d\mu(t), \quad \mu *_c \nu := \frac{1}{2}(\mu * \nu + \mu \circ \nu + \nu \circ \mu).$$

The proof of the following proposition is straightforward.

Proposition 2.5: *Let ω be an extendible weight function on $\mathbb{R}^+$ with growth constant C, $f \in L^1_\omega(\mathbb{R}^+)$ and $\mu, \nu \in M_\omega(\mathbb{R}^+)$. Then:*
(i) $\mu \circ \nu, \mu *_c \nu \in M_\omega(\mathbb{R}^+)$;
(ii) $\|\mu *_c \nu\|_\omega \leq C\|\mu\|_\omega\|\nu\|_\omega$;
(iii) $(\mu *_c \nu) *_c f = \mu *_c (\nu *_c f)$.

Let ω be an extendible weight function on $\mathbb{R}^+$ with growth constant C. Then, when endowed with $*_c$, the Banach space $M_\omega(\mathbb{R}^+)$ is in fact a Banach algebra. We denote it by $M_\omega(\mathbb{R}^+, *_c)$. Let us define the map $T : M_\omega(\mathbb{R}^+, *_c) \to Mul(L^1_\omega(\mathbb{R}^+, *_c))$, $\mu \mapsto T_\mu$, by

$$T_\mu(f) := \mu *_c f, \quad f \in L^1_\omega(\mathbb{R}^+, *_c).$$

Since $\|\mu *_c f\|_\omega \le C\|\mu\|_\omega \|f\|_\omega$ (Proposition 2.2 (iv)) and $\mu *_c (f *_c g) = f *_c (\mu *_c g)$ for $f, g \in L^1_\omega(\mathbb{R}^+, *_c)$ (Proposition 2.2 (ii)), it follows that T_μ is a multiplier on $L^1_\omega(\mathbb{R}^+, *_c)$, $\|T_\mu\| \le C\|\mu\|_\omega$ and $\|T\| \le C$. Moreover, using Proposition 2.5 (iii), the map T is an algebra homomorphism: if $\mu, \nu \in M_\omega(\mathbb{R}^+)$ and $f \in L^1_\omega(\mathbb{R}^+)$, then

$$T_{\mu *_c \nu}(f) = (\mu *_c \nu) *_c f = \mu *_c (\nu *_c f) = T_\mu(T_\nu(f)).$$

THEOREM 2.6: *The map T is a bounded algebra isomorphism from $M_\omega(\mathbb{R}^+, *_c)$ onto $Mul(L^1_\omega(\mathbb{R}^+, *_c))$.*

Proof. It is enough to show that each $S \in Mul(L^1_\omega(\mathbb{R}^+, *_c))$ is equal to T_μ for some $\mu \in M_\omega(\mathbb{R}^+)$ such that $\|\mu\|_\omega \le M\|S\|$ for some $M > 0$.

Since $L^1_\omega(\mathbb{R}^+, *_c)$ has a bounded approximate identity (Proposition 1.4), for each $S \in Mul(L^1_\omega(\mathbb{R}^+, *_c))$ there exists a net $\{s_\beta\}_{\beta \in B} \subset L^1_\omega(\mathbb{R}^+, *_c)$ such that

$$S(f) = \lim_{\beta \in B} (s_\beta *_c f), \quad f \in L^1_\omega(\mathbb{R}^+, *_c), \tag{2.1}$$

and $\sup_{\beta \in B} \|s_\beta\|_\omega \le M\|S\|$ (cf. [4, Proposition 2.1]).

Note that $\{s_\beta\}_{\beta \in B} \subset (C_{0,\omega}(\mathbb{R}^+))'$ and by the relative weak* compactness of bounded sets in $(C_{0,\omega}(\mathbb{R}^+))'$, there exists $\mu \in M_\omega(\mathbb{R}^+)$ and a subnet (which we write in the same way), $\{s_\beta\}_{\beta \in B}$, such that

$$\lim_{\beta \in B} \int_0^\infty g(t) s_\beta(t) dt = \int_0^\infty g(t) d\mu(t), \quad g \in C_{0,\omega}(\mathbb{R}^+), \tag{2.2}$$

and $\|\mu\| \le M\|S\|$. Take $f \in L^1_\omega(\mathbb{R}^+)$ and $\psi \in C_{0,\omega}(\mathbb{R}^+)$. Then $\psi *_c f \in C_{0,\omega}(\mathbb{R}^+)$ by assertion (i) of Lemma 2.3, and

$$\int_0^\infty (\psi *_c f)(t) s_\beta(t) dt = \int_0^\infty \psi(t)(s_\beta *_c f)(t) dt,$$

by assertion (ii) of Lemma 2.3. Using the same ideas, it is easy to check that

$$\int_0^\infty (\psi *_c f)(t) d\mu(t) = \int_0^\infty \psi(t)(\mu *_c f)(t) dt.$$

Now we can write (2.2) as

$$\int_0^\infty \psi(t)(\mu *_c f)(t)dt = \lim_{\beta\in B} \int_0^\infty \psi(t)(s_\beta *_c f)(t)dt = \int_0^\infty \psi(t)S(f)(t)dt$$

where we apply (2.1). Since ψ is an arbitrary element of $C_{0,\omega}(\mathbb{R}^+)$, we have $S(f) = f *_c \mu$. ∎

Remark: The above proof was inspired by the proof of [4, Theorem 3.2]. In fact, Chojnacki's theorem is an adaptation of a result of J. G. Wendel on group algebras, see more comments and details in [4].

3. A variant of Kisyński theorem on $L^1_\omega(\mathbb{R}^+, *_c)$

Let ω be an extendible weight function with bound κ, $\mathcal{A}$ a commutative Banach algebra and $r : (\kappa, \infty) \to \mathcal{A}$ a function. By Theorem 0.1, there exists $T : L^1_\omega(\mathbb{R}^+) \to \mathcal{A}$ such that $r(\lambda) = T(\epsilon_{-\lambda})$ for $\lambda > \kappa$ if and only if $r \in C^{(\infty)}((\kappa,\infty), \mathcal{A})$ and

$$\|r\|_{W,\omega,\kappa} < \infty.$$

Next we prove a result similar to Theorem 0.2 for the algebra $L^1_\omega(\mathbb{R}^+, *_c)$. For $\omega(t) = 1$ this results appears in [2, Proposition 5.1].

THEOREM 3.1: *Let $\mathcal{A}$ be a Banach algebra, ω be an extendible weight function on $\mathbb{R}^+$ with bound $\kappa \geq 0$, $r \in C^{(\infty)}((\kappa,\infty), \mathcal{A})$ such that*

$$\|r\|_{W,\omega,\kappa} < \infty.$$

Then the following conditions are equivalent:

(i) *The bounded homomorphism $T : L^1_\omega(\mathbb{R}^+) \to \mathcal{A}$ such that $r(\lambda) = T(\epsilon_{-\lambda})$ for $\lambda > \kappa$ is an algebra homomorphism $T : L^1_\omega(\mathbb{R}^+, *_c) \to \mathcal{A}$.*

(ii) *The function $(r(\sqrt{\lambda})/\sqrt{\lambda})_{\lambda > \kappa^2}$ is a pseudo-resolvent on $\mathcal{A}$.*

Proof. We define $R(\lambda) := r(\sqrt{\lambda})/\sqrt{\lambda}$ for $\lambda > \kappa^2$. (i) ⇒ (ii) If $\lambda, \nu > \kappa^2$, then

$$\begin{aligned} R(\lambda)R(\nu) &= \frac{1}{\sqrt{\lambda}\sqrt{\nu}} T(\epsilon_{-\sqrt{\lambda}})T(\epsilon_{-\sqrt{\nu}}) = \frac{1}{\sqrt{\lambda}\sqrt{\nu}} T(\epsilon_{-\sqrt{\lambda}} *_c \epsilon_{-\sqrt{\nu}}) \\ &= \frac{1}{\sqrt{\lambda}\sqrt{\nu}} T\Big(\frac{1}{\lambda-\nu}\big(\sqrt{\lambda}\epsilon_{-\sqrt{\nu}} - \sqrt{\nu}\epsilon_{-\sqrt{\lambda}}\big)\Big) \\ &= \frac{1}{\lambda-\nu}\Big(\frac{1}{\sqrt{\nu}}T(\epsilon_{-\sqrt{\nu}}) - \frac{1}{\sqrt{\lambda}}T(\epsilon_{-\sqrt{\lambda}})\Big) = \frac{1}{\lambda-\nu}\left(R(\nu) - R(\lambda)\right), \end{aligned}$$

where we have used the equality (1.4).

(ii) ⇒ (i) Since the linear space spanned by the set $\{\epsilon_{-\nu}\colon \nu > \kappa\}$ is dense in $L^1_\omega(\mathbb{R}^+)$ ([5, Proposition 2.2]) and T is linear and bounded, it is enough to check that

$$T(\epsilon_{-\lambda} *_c \epsilon_{-\nu}) = T(\epsilon_{-\lambda})T(\epsilon_{-\nu}),$$

for $\lambda, \nu > \kappa$. Since $(R(\lambda))_{\lambda>\kappa^2}$ is a pseudo-resolvent, we have

$$\begin{aligned} T(\epsilon_{-\lambda})T(\epsilon_{-\nu}) &= \lambda\mu R(\lambda^2)R(\nu^2) = \frac{\lambda\nu}{\lambda^2-\nu^2}\left(R(\nu^2)-R(\lambda^2)\right) \\ &= \frac{1}{\lambda^2-\nu^2}\left(\lambda r(\nu) - \nu r(\lambda)\right) = T(\epsilon_{-\lambda} *_c \epsilon_{-\nu}), \end{aligned}$$

where we have used again the equality (1.4) for $\lambda, \nu > \kappa$. ∎

In Example 3, Section 1, we defined the Gaussian semigroup in $L^1_\omega(\mathbb{R}^+, *_c)$. We now consider the Gaussian transform $\mathcal{G} : L^1_{\omega_g}(\mathbb{R}^+, *) \to L^1_\omega(\mathbb{R}^+, *_c)$ given by

$$\mathcal{G}(f)(s) := \int_0^\infty f(t)g^t(s)dt, \quad s \in \mathbb{R}^+,\ f \in L^1_{\omega_g}(\mathbb{R}^+, *),$$

where $\omega_g(t) := \|g^t\|_\omega$ for $t \geq 0$. Then

$$\mathcal{G}(\epsilon_{-\lambda}) = \frac{1}{\sqrt{\lambda}}\epsilon_{-\sqrt{\lambda}}, \quad \lambda > \kappa^2, \tag{3.1}$$

(cf. [15]) and $\mathcal{G}$ is an algebra homomorphism with $\|\mathcal{G}\| \leq 1$.

Let X be a Banach space and $T \in \mathcal{B}(L^1_\omega(\mathbb{R}^+), X)$. The Gaussian transform allows one to define $T_\mathcal{G} \in \mathcal{B}(L^1_{\omega_g}(\mathbb{R}^+), X)$ by

$$T_\mathcal{G}(f) := T(\mathcal{G}(f)), \quad f \in L^1_{\omega_g}(\mathbb{R}^+).$$

Corollary 3.2: *Let $\mathcal{A}$ be a Banach algebra, ω an extendible weight function on $\mathbb{R}^+$ with bound $\kappa \geq 0$ and $r \in C^{(\infty)}((\kappa,\infty), \mathcal{A})$ such that*

$$\|r\|_{W,\omega,\kappa} < \infty.$$

Then the following conditions are equivalent:

(i) *The bounded homomorphism $T : L^1_\omega(\mathbb{R}^+) \to \mathcal{A}$ is an algebra homomorphism, $T : L^1_\omega(\mathbb{R}^+, *_c) \to \mathcal{A}$, with*

$$r(\lambda) = T(\epsilon_{-\lambda}), \quad \lambda > \kappa.$$

(ii) *The bounded homomorphism $T_\mathcal{G} : L^1_{\omega_g}(\mathbb{R}^+) \to \mathcal{A}$ is an algebra homomorphism, $T_\mathcal{G} : L^1_{\omega_g}(\mathbb{R}^+, *) \to \mathcal{A}$, such that*

$$T_\mathcal{G}(\epsilon_{-\lambda}) = \frac{r(\sqrt{\lambda})}{\sqrt{\lambda}}, \quad \lambda > \kappa^2.$$

(iii) *The function* $(r(\sqrt{\lambda})/\sqrt{\lambda})_{\lambda>\kappa^2}$ *is a pseudo-resolvent on* $\mathcal{A}$.

Proof. The implication (i)⇒(ii) follows from the identities

$$T_{\mathcal{G}}(f*g) = T(\mathcal{G}(f*g)) = T(\mathcal{G}(f) *_c \mathcal{G}(g)) = T_{\mathcal{G}}(f)T_{\mathcal{G}}(g), \quad f, g \in L^1_{\omega_g}(\mathbb{R}^+, *).$$

The implication (ii)⇒(iii) is part of Theorem 0.2 and (iii)⇒(i) is part of Theorem 3.1. ∎

4. Algebra homomorphims and integrated cosine functions

In this section we characterize bounded algebra homomorphims from $L^1_\omega(\mathbb{R}^+, *_c)$ in terms of integrated cosine functions (Theorem 4.8). To show this we use certain fractional Banach algebras $\mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c)$ which have been introduced in [11]. We also need to give an extension of a result of [12] about uniformly bounded limits of fractional homomorphisms (Theorem 4.6).

Let $\mathcal{D}_+$ denote the set of test functions of compact support in $[0,\infty)$, and $\mathcal{S}_+$ denote the Schwartz class on $[0,\infty)$, i.e., functions that are infinitely differentiable and satisfy

$$\sup_{t\geq 0}\left|t^m \frac{d^n}{dt^n} f(t)\right| < \infty,$$

for any $m, n \in \mathbb{N}\cup\{0\}$.

Given $f \in \mathcal{S}_+$, the **Weyl fractional integral** of f of order $\alpha > 0$ is defined by

$$W_+^{-\alpha} f(u) := \frac{1}{\Gamma(\alpha)}\int_u^\infty (t-u)^{\alpha-1} f(t)dt, \quad u \geq 0.$$

This operator $W_+^{-\alpha} : \mathcal{S}_+ \to \mathcal{S}_+$ is one to one, its inverse, W_+^α, is the **Weyl fractional derivative** of order α and

$$W_+^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^\infty (s-t)^{n-\alpha-1} f(s)ds, \quad t \geq 0,$$

holds with $n = [\alpha]+1$, see for example [14]. It is easy to check that $W_+^\alpha f = (-1)^\alpha f^{(\alpha)}$ if $\alpha \in \mathbb{N}$. The semigroup law $W_+^{\alpha+\beta} f = W_+^\alpha(W_+^\beta f)$ holds with $\alpha, \beta \in \mathbb{R}$, $W_+^0 = \mathrm{Id}$, and $W_+^\alpha(f_\lambda)(t) = \lambda^\alpha W_+^\alpha(f)(\lambda t)$ with $\lambda > 0$, if $f_\lambda(t) := f(\lambda t)$ and $f \in \mathcal{S}_+$ (cf. [14]).

Example 4: If $\lambda > 0$, then $\epsilon_{-\lambda} \in \mathcal{S}_+$ and

$$W_+^{-\alpha}(\epsilon_{-\lambda}) = \lambda^{-\alpha}\epsilon_{-\lambda}.$$

Therefore, $W_+^\alpha \epsilon_{-\lambda} = \lambda^\alpha \epsilon_{-\lambda}$ for $\alpha \in \mathbb{R}$. We define functions $(\beta_{n,\lambda})_{n\in\mathbb{N},\lambda>0}$ by

$$\beta_{n,\lambda}(t) := t^n e^{-\lambda t}, \quad t \geq 0.$$

Note that $\beta_{n,\lambda}(t) = (-1)^n (d/d\lambda)^n \epsilon_{-\lambda}(t)$ for $t, \lambda > 0$ and $n \in \mathbb{N} \cup \{0\}$. To give the value of $W_+^\alpha(\beta_{n,\lambda})$ in Theorem 4.2, we show the following lemma.

LEMMA 4.1: *If $\alpha \in \mathbb{R}$ and $f \in \mathcal{S}_+$, then*

$$W_+^\alpha(sf(s))(t) = tW_+^\alpha f(t) - \alpha W_+^{\alpha-1} f(t), \quad t > 0.$$

Proof. The case $\alpha < 0$ is shown in [14, p. 246]; if $\alpha > 0$ we have

$$W_+^{-\alpha}(sW_+^\alpha f(s) - \alpha W_+^{\alpha-1} f(s))(t) = tf(t) + \alpha W_+^{-1} f(t) - \alpha W_+^{-1} f(t) = tf(t)$$

with $t > 0$. ∎

Polynomial solutions of the differential equation

$$zy''(z) + (\alpha + 1 - z)y'(z) + ny(z) = 0,$$

with $n = 0, 1, 2 \ldots$ and $\alpha \in \mathbb{C}$, are called **generalized Laguerre polynomials,** and are denoted by $L_n^{(\alpha)}$. They satisfy a Rodrigues' formula,

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n}(x^{n+\alpha} e^{-x}), \tag{4.1}$$

see [10, p. 241], and we have

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n} (-1)^m \binom{n+\alpha}{n-m} \frac{x^m}{m!}, \quad x \in \mathbb{R},$$

where $\binom{n+\alpha}{n-m} = \frac{(\alpha+m+1)_{n-m}}{(n-m)!}$ and $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$.

Generalized Laguerre polynomials satisfy several recurrence equalities, see [10, p. 241], one of them being

$$xL_n^{(\alpha+1)}(x) = (n+\alpha+1)L_n^{(\alpha)}(x) - (n+1)L_{n+1}^{(\alpha)}(x), \quad x \in \mathbb{R}. \tag{4.2}$$

THEOREM 4.2: *Let $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$ and $\lambda > 0$. Then*

$$W_+^\alpha(\beta_{n,\lambda})(t) = \lambda^{\alpha-n} e^{-\lambda t} (-1)^n n! L_n^{(\alpha-n)}(\lambda t), \quad t \geq 0.$$

Proof. Note that it is enough to show that

$$L_n^{(\alpha-n)}(x) = \frac{(-1)^n}{n!} e^x W_+^\alpha(t^n e^{-t})(x), \quad x \geq 0.$$

We prove this by induction on n. Take $\alpha > 0$; for $n = 1$ we apply Lemma 4.1 to get

$$W_+^\alpha(te^{-t})(x) = xW_+^\alpha(e^{-t})(x) - \alpha W_+^{\alpha-1}(e^{-t})(x) = e^{-x}(x-\alpha) = -L_1^{(\alpha-1)}(x).$$

Consider the case $n+1$. By Lemma 4.1 we obtain that

$$W_+^\alpha(t^{n+1}e^{-t})(x) = xW_+^\alpha(t^n e^{-t})(x) - \alpha W_+^{\alpha-1}(t^n e^{-t})(x), \quad x \geq 0,$$

and by the induction hypothesis,

$$W_+^\alpha(t^{n+1}e^{-t})(x) = e^{-x} n!(-1)^n \left(xL_n^{(\alpha-n)}(x) - \alpha L_{n+1}^{(\alpha-1-n)}(x) \right), \quad x \geq 0.$$

Then we apply the recurrence formula (4.2) to derive that

$$\begin{aligned} W_+^\alpha(t^{n+1}e^{-t})(x) &= e^{-x} n!(-1)^n(-1)(n+1)L_{n+1}^{(\alpha-n-1)}(x) \\ &= e^{-x}(-1)^{n+1}(n+1)!L_{n+1}^{(\alpha-(n+1))}(x) \quad x \geq 0. \end{aligned}$$

The case $\alpha < 0$ can be proved in a similar way. ∎

In the remainder of this section we consider a continuous non-decreasing weight $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ with $\omega(0) \neq 0$ and bound $\kappa \geq 0$. Then ω is an extendible weight due to the fact that

$$\omega(t-s) \leq \omega(t) \leq \omega(t)\frac{\omega(s)}{\omega(0)} = C\omega(t)\omega(s), \quad 0 \leq s \leq t.$$

Let $\tau_\alpha(t) := t^\alpha \omega(t)$, $\alpha > 0$ and let $(R_t^\theta)_{t>0}$ be the family of Riesz functions given by

$$R_t^\theta(s) := \frac{(t-s)^\theta}{\Gamma(\theta+1)}\chi_{(0,t)}(s),$$

with $t, s > 0$ and $\theta > -1$.

Theorem 4.3 ([11, Theorem 3]): *Let $\alpha > 0$. Then the expression*

$$q_{\tau_\alpha}(f) := \frac{1}{\Gamma(\alpha+1)} \int_0^\infty \tau_\alpha(t)|W_+^\alpha f(t)|dt, \quad f \in \mathcal{D}_+,$$

defines a norm on $\mathcal{D}_+$. We denote by $\mathcal{T}_+^{(\alpha)}(\tau_\alpha)$ the Banach space obtained as the completion of $\mathcal{D}_+$ in the norm q_{τ_α}. The following assertions hold:

(i) $\mathcal{T}_+^{(\alpha)}(\tau_\alpha) \hookrightarrow \mathcal{T}_+^{(\alpha)}(t^\alpha) \hookrightarrow L^1(\mathbb{R}^+)$ *and* $\mathcal{T}_+^{(\alpha)}(\tau_\alpha) \hookrightarrow L_\omega^1(\mathbb{R}^+)$.

(ii) $\mathcal{T}_+^{(\beta)}(\tau_\beta) \hookrightarrow \mathcal{T}_+^{(\alpha)}(\tau_\alpha)$; *in particular,* $\mathcal{T}_+^{(\beta)}(t^\beta) \hookrightarrow \mathcal{T}_+^{(\alpha)}(t^\alpha)$.

(iii) $R_t^{\nu-1} \in \mathcal{T}_+^{(\alpha)}(\tau_\alpha)$ *with* $t > 0$ *and* $\nu > \alpha$; *and* $q_{\tau_\alpha}(R_t^{\nu-1}) \leq C_{\nu,\alpha} t^{\nu-\alpha}\tau_\alpha(t)$ *for* $t > 0$, *where* $C_{\nu,\alpha} > 0$ *is independent of* t.

Moreover, $q_{\tau_\alpha}(f *_c g) \le C_\alpha q_{\tau_\alpha}(f) q_{\tau_\alpha}(g)$ *for* $f, g \in \mathcal{D}_+$, *and* $C_\alpha > 0$ *is independent of* f *and* g. *We denote by* $\mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c)$ *the Banach algebra obtained as the completion of* $\mathcal{D}_+$ *in the norm* q_{τ_α} *with the cosine convolution product* $*_c$.

Example 5: If $\lambda > \kappa$, then $\epsilon_{-\lambda} \in \mathcal{T}_+^{(\alpha)}(\tau_\alpha)$ and

$$q_{\tau_\alpha}(\epsilon_{-\lambda}) \le \lambda^\alpha / (\lambda - \kappa)^{\alpha+1}.$$

Since $\beta_{n,\lambda} = n! \underbrace{\epsilon_{-\lambda} * \cdots * \epsilon_{-\lambda}}_{n+1 \text{ times}}$ for $n \in \mathbb{N} \cup \{0\}$ and $\lambda > 0$, we obtain $\beta_{n,\lambda} \in \mathcal{T}_+^{(\alpha)}(\tau_\alpha)$ for $n \in \mathbb{N} \cup \{0\}$ and $\lambda > \kappa$.

Example 6: Solutions of the **Hermite differential equation**

$$w''(z) - 2zw'(z) + 2\nu w(z) = 0,$$

with $\nu \in \mathbb{C}$ are called **Hermite functions** of order ν, and are denoted H_ν. For $\nu \in \mathbb{N}$, the functions H_ν are polynomials known as Hermite polynomials. They satisfy a Rodrigues' formula

$$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2}.$$

Moreover, for $\nu \in \mathbb{R}$, some Hermite functions satisfy that

$$W_+^\nu(e^{-t^2})(t) = e^{-t^2} H_\nu(t), \quad t \in \mathbb{R},$$

and

$$|H_\nu(z)| \le C_\nu(1 + |z|^\nu), \quad \nu > 0,\ \Re z > 0, \tag{4.3}$$

see for example [8, p. 344].

Proposition 4.4: *Let* $(g^z)_{\Re z > 0}$ *be the Gaussian semigroup. Then* $(g^z)_{\Re z>0} \subset \mathcal{T}_+^{(\alpha)}(\tau_\alpha)$ *and*

$$q_{\tau_\alpha}(g^z) \le C_\alpha e^{\kappa^2 |z|^2/\Re z}\Big(\Big(\frac{|z|}{\Re z}\Big)^{\alpha+1/2} + \kappa^\alpha \frac{|z|^{3/2\alpha+1/2}}{(\Re z)^{\alpha+1/2}} + \kappa^{2\alpha}\frac{|z|^{3\alpha+1/2}}{(\Re z)^{2\alpha+1/2}}\Big),\ \Re z > 0,$$

where $\tau_\alpha(t) \le Ct^\alpha e^{\kappa t}$ *for* $\kappa, t \ge 0$.

Proof. Note that for $\alpha, z > 0$,

$$W_+^\alpha(g^z)(t) = \frac{1}{\sqrt{\pi z}} W_+^\alpha(e^{-(\frac{r}{2\sqrt{z}})^2})(t) = \frac{1}{2^\alpha \sqrt{\pi} z^{(\alpha+1)/2}} H_\alpha\Big(\frac{t}{2\sqrt{z}}\Big) e^{-t^2/(4z)},$$

for $t > 0$ and then for $z \in \mathbb{C}^+$, because of the analytic continuation principle. Now we apply (4.3) to obtain

$$\begin{aligned} q_{\tau_\alpha}(g^z) &\leq \frac{1}{2^\alpha \sqrt{\pi}|z|^{(\alpha+1)/2}} \int_0^\infty |H_\alpha(t/(2\sqrt{z}))| e^{-t^2 \Re z/4|z|^2} \tau_\alpha(t) dt \\ &\leq \frac{C_\alpha}{|z|^{(\alpha+1)/2}} \int_0^\infty \Big(1 + \frac{t^\alpha}{2^\alpha |z|^{\alpha/2}}\Big) e^{-t^2 \Re z/4|z|^2} t^\alpha e^{\kappa t} dt. \end{aligned}$$

We change the variable $u = t\sqrt{\Re z}/(2|z|) - (\kappa|z|)/\sqrt{\Re z}$, and get

$$q_{\tau_\alpha}(g^z) \leq C_\alpha e^{\kappa^2 |z|^2/\Re z} \Big(\Big(\frac{|z|}{\Re z}\Big)^{\alpha+1/2} + \kappa^\alpha \frac{|z|^{3/2\alpha+1/2}}{(\Re z)^{\alpha+1/2}} + \kappa^{2\alpha} \frac{|z|^{3\alpha+1/2}}{(\Re z)^{2\alpha+1/2}} \Big),$$

for $\Re z > 0$. ∎

Lemma 4.5: *The function $(\kappa, \infty) \to \mathcal{T}_+^{(\alpha)}(\tau_\alpha)$, $\lambda \mapsto \epsilon_{-\lambda}$ is infinitely differentiable in the norm topology of $\mathcal{T}_+^{(\alpha)}(\tau_\alpha)$ and*

$$\frac{d^n \epsilon_{-\lambda}}{d\lambda^n} = (-1)^n \beta_{n,\lambda},$$

for each $n \in \mathbb{N} \cup \{0\}$ and $\lambda > \kappa$.

Proof. Take $\lambda > \kappa$. Then

$$\beta_{n,\lambda}(t) = (-1)^n \frac{d^n}{d\lambda^n} \epsilon_{-\lambda}(t) = (-1)^n \int_0^\infty \frac{d^n}{d\lambda^n} \Big(\lambda e^{-\lambda^2 s}\Big)(\lambda) g^s(t) ds, \quad t \geq 0,$$

where we use the equality (3.1). Note that

$$\int_0^\infty \Big| \frac{d^n}{d\lambda^n} \Big(\lambda e^{-\lambda^2 s}\Big)(\lambda) \Big| q_{\tau_\alpha}(g^s) ds < \infty,$$

by Proposition 4.4 and we obtain that

$$\beta_{n,\lambda} = (-1)^n \int_0^\infty \frac{d^n}{d\lambda^n} \Big(\lambda e^{-\lambda^2 s}\Big)(\lambda) g^s ds = (-1)^n \frac{d^n}{d\lambda^n} \epsilon_{-\lambda},$$

in the norm topology of $\mathcal{T}_+^{(\alpha)}(\tau_\alpha)$. ∎

The next result is a generalization of [12, Theorem 3.1] concerning bounded limits of fractional homomorphisms.

Theorem 4.6: *Let X be a Banach space, ω a continuous non-decreasing weight with $\omega(0) \neq 0$ and bound $\kappa \geq 0$; and $\tau_\alpha(t) := t^\alpha \omega(t)$ for $t \geq 0$ and $\alpha \geq 0$.*

(i) *For every bounded homomorphism $T : L^1_\omega(\mathbb{R}^+) \to X$, there exists a family of bounded homomorphisms $T_\alpha : \mathcal{T}_+^{(\alpha)}(\tau_\alpha) \to X$, $\alpha > 0$, such that $\|T_\alpha\| \leq \|T\|$ and $T_\alpha(\epsilon_{-\lambda}) = T(\epsilon_{-\lambda})$ for each $\lambda \in (\kappa, \infty)$.*

(ii) *Conversely, if for each* $\alpha > 0$ *there exists a bounded homomorphism* $T_\alpha : \mathcal{T}_+^{(\alpha)}(\tau_\alpha) \to X$ *such that* $T_\alpha(\epsilon_{-\lambda})$ *does not depend on* α *for each* $\lambda \in (\omega, \infty)$ *and* $\limsup_{\alpha\to 0^+} \|T_\alpha\| < \infty$, *then there exists a unique bounded homomorphism* $T : L^1_\omega(\mathbb{R}^+) \to X$ *such that* $T(\epsilon_{-\lambda}) = T_\alpha(\epsilon_{-\lambda})$ *for each* $\lambda \in (\kappa, \infty)$, $\alpha > 0$ *and* $\|T\| \le \limsup_{\alpha\to 0^+} \|T_\alpha\|$.

Proof. The first part follows from $\mathcal{T}_+^{(\alpha)}(\tau_\alpha) \hookrightarrow L^1_\omega(\mathbb{R}^+)$, see part (i) of Theorem 4.3. To prove (ii), we use Theorem 0.1. We define $r(\lambda) := T_\alpha(\epsilon_{-\lambda})$ for each $\lambda > \kappa$. The family $(r(\lambda))_{\lambda\in(\omega,\infty)}$ is well-defined. We apply Lemma 4.5 to obtain $r^{(n)}(\lambda) = (-1)^n T_\alpha(\beta_{n,\lambda})$ and Theorem 4.2 to conclude that

$$\|r^{(n)}(\lambda)\| \le \|T_\alpha\|\ q_{\tau_\alpha}(\beta_{n,\lambda}) = \frac{\|T_\alpha\| n!}{\Gamma(\alpha+1)}\lambda^{\alpha-n}\int_0^\infty e^{-\lambda t}|L_n^{(\alpha-n)}(\lambda t)|t^\alpha\omega(t)dt,$$

for each $n \in \mathbb{N}$, $\lambda \in (\kappa, \infty)$ and $\alpha > 0$. Since

$$\lim_{\alpha\to 0^+} L_n^{(\alpha-n)}(\lambda t) = (-1)^n(\lambda t)^n/n!,$$

we have

$$\|r^{(n)}(\lambda)\| \le \limsup_{\alpha\to 0^+}\|T_\alpha\|\int_0^\infty t^n e^{-\lambda t}\omega(t)dt = \limsup_{\alpha\to 0^+}\|T_\alpha\|\,\|\beta_{n,\lambda}\|_\omega,$$

for each $n \in \mathbb{N}$ and $\lambda \in (\kappa, \infty)$. By Theorem 0.1 there exists a unique bounded homomorphism $T : L^1_\omega(\mathbb{R}^+) \to X$ such that $T(\epsilon_{-\lambda}) = r(\lambda) = T_\alpha(\epsilon_{-\lambda})$ for each $\lambda \in (\kappa, \infty)$ and $\|T\| \le \limsup_{\alpha\to 0^+}\|T_\alpha\|$. ∎

To finish the section we characterize bounded algebra homomorphisms from $L^1_\omega(\mathbb{R}^+, *_c)$ into a Banach algebra $\mathcal{A}$. The case of $L^1_\omega(\mathbb{R}^+, *)$ has been considered in detail in several papers ([3], [4, Theorem 3.3], [9, Section 10]). In our setting, cosine functions need to be considered.

Definition 4.7: For any $\alpha > 0$, an **α-times integrated cosine function** is a continuous mapping $c_\alpha(\,\cdot\,) : [0, \infty) \to \mathcal{A}$ such that $c_\alpha(0) = 0$ and satisfying

$$2\Gamma(\alpha)c_\alpha(t)c_\alpha(s) = \int_t^{t+s}(t+s-r)^{\alpha-1}c_\alpha(r)dr - \int_0^s (t+s-r)^{\alpha-1}c_\alpha(r)dr$$

$$+\int_{t-s}^t (r-t+s)^{\alpha-1}c_\alpha(r)dr + \int_0^s (r+t-s)^{\alpha-1}c_\alpha(r)dr \tag{4.4}$$

whenever $t > s > 0$.

If $(c_\alpha(t))_{t\geq 0}$ is an α-times integrated cosine function in $\mathcal{A}$, then $(c_\nu(t))_{t\geq 0}$, where

$$c_\nu(t) := \frac{1}{\Gamma(\nu-\alpha)} \int_0^t (t-s)^{\nu-\alpha-1} c_\alpha(s) ds, \quad t \geq 0,$$

is a ν-times integrated cosine function in $\mathcal{A}$ for every $\nu > \alpha$. The set of Riesz functions $(R_t^{\nu-1})_{t\geq 0}$ is an example of ν-times integrated cosine function in $\mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c)$ for $\nu > \alpha \geq 0$, see [11].

The following result is inspired by [12, Theorem 4.2].

THEOREM 4.8: *Let $\mathcal{A}$ be a Banach algebra, ω a continuous non-decreasing weight with $\omega(0) \neq 0$ and bound $\kappa \geq 0$. Let $\tau_\alpha(t) := t^\alpha \omega(t)$ for $t, \alpha \geq 0$ and $r \in C^{(\infty)}((\kappa,\infty),\mathcal{A})$ such that*

$$M = \sup\left\{ \frac{\|r^{(k)}(\lambda)\|}{\|t^k e^{-\lambda t}\|_\omega} : k \in \mathbb{N} \cup \{0\}, \lambda \in (\kappa,\infty)\right\}.$$

Then the following conditions are equivalent:

(i) *$M < \infty$ and $(r(\sqrt{\lambda})/\sqrt{\lambda})_{\lambda > \kappa^2}$ is a pseudo-resolvent on $\mathcal{A}$.*

(ii) *There exists a bounded algebra homomorphism $T : L^1_\omega(\mathbb{R}^+, *_c) \to \mathcal{A}$ such that $T(\epsilon_{-\lambda}) = r(\lambda)$ for each $\lambda \in (\kappa,\infty)$.*

(iii) *For any $\alpha > 0$, there exists an α-times integrated cosine function $(c_\alpha(t))_{t\geq 0}$ in $\mathcal{A}$ such that $\|c_\alpha(t)\| \leq \frac{C}{\Gamma(\alpha+1)}\tau_\alpha(t)$ for some constant $C > 0$ and every $t \geq 0$, and $r(\lambda) = \lambda^\alpha \int_0^\infty e^{-\lambda t} c_\alpha(t) dt$ for $\lambda > \kappa$.*

(iv) *For any $\alpha > 0$, there exists a bounded algebra homomorphism $T_\alpha : \mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c) \to \mathcal{A}$ such that $T_\alpha(\epsilon_{-\lambda}) = r(\lambda)$ for each $\lambda \in (\kappa,\infty)$ and $\sup_{\alpha>0} \|T_\alpha\| < \infty$.*

Furthermore, if there exists a bounded algebra homomorphism

$$T : L^1_\omega(\mathbb{R}^+, *_c) \to \mathcal{A}$$

such that $T(\epsilon_{-\lambda}) = r(\lambda)$ for each $\lambda \in (\kappa,\infty)$, then it is unique, $T(f) = T_\alpha(f)$ for $f \in \mathcal{T}_+^{(\alpha)}(\tau_\alpha)$ and every $\alpha > 0$ and

$$M = \|T\| = \sup_{\alpha>0} \|T_\alpha\| = \inf\left\{ C : \|c_\alpha(t)\| \leq C\tau_\alpha(t)/\Gamma(\alpha+1),\ t \geq 0\right\}.$$

Proof. (i) $\Leftrightarrow$ (ii) is the content of Theorem 3.1. (ii) $\Rightarrow$ (iii) Noting that, for each $t > 0$, the Riesz function $R_t^{\alpha-1}$ is a member of $L^1_\omega(\mathbb{R}^+, *_c)$, we define $c_\alpha(t) := T(R_t^{\alpha-1})$ for $t > 0$ and $c_\alpha(0) := 0$. Since $(R_t^{\alpha-1})_{t>0}$ is an α-times integrated cosine function in $L^1_\omega(\mathbb{R}^+, *_c)$, $(c_\alpha(t))_{t\geq 0}$ is an α-times integrated cosine function in $\mathcal{A}$ and

$$\|c_\alpha(t)\| \le \|T\|\|R_t^{\alpha-1}\|_\omega \le \|T\|\frac{t^\alpha}{\Gamma(\alpha+1)}\omega(t),$$

for $t \ge 0$. By the continuity of T, we have

$$\lambda^\alpha \int_0^\infty e^{-\lambda t} c_\alpha(t)dt = \lambda^\alpha T\Big(\int_0^\infty e^{-\lambda t} R_t^{\alpha-1} dt\Big) = \lambda^\alpha T(W^{-\alpha}\epsilon_{-\lambda})$$
$$= T(\epsilon_{-\lambda}) = r(\lambda)$$

for $\lambda > \omega$. (iii) $\Rightarrow$ (iv) We define $T_\alpha : \mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c) \to \mathcal{A}$ by

$$T_\alpha(f) := \int_0^\infty W_+^\alpha f(t) c_\alpha(t)dt, \quad f \in \mathcal{D}_+.$$

Following the same arguments as in [11, Theorem 4], we prove that T_α is a bounded algebra homomorphism with $\|T_\alpha\| \le C < \infty$ for any $\alpha > 0$. Since $\epsilon_{-\lambda} \in \mathcal{T}_+^{(\alpha)}(\tau_\alpha)$ for $\lambda > \kappa$ and $W_+^\alpha(\epsilon_{-\lambda}) = \lambda^\alpha \epsilon_{-\lambda}$, we have $T_\alpha(\epsilon_{-\lambda}) = r(\lambda)$ for $\lambda > \kappa$.

The implication (iv) $\Rightarrow$ (ii) follows from assertion (ii) of Theorem 4.6.

Finally, if there exists a bounded algebra homomorphism $T : L^1_\omega(\mathbb{R}^+, *_c) \to \mathcal{A}$ satisfying $T(\epsilon_{-\lambda}) = r(\lambda)$ for each $\lambda \in (\omega, \infty)$, then $\mathcal{T}_+^{(\alpha)}(\tau_\alpha) \hookrightarrow L^1_\omega(\mathbb{R}^+)$ and $T(f) = T_\alpha(f)$ for $f \in \mathcal{T}_+^{(\alpha)}(\tau_\alpha)$ and for all $\alpha > 0$. Moreover, we have collected in the proof the following inequalities,

$$\|T\| = M = \sup_{\alpha>0} \|T_\alpha\| \le \inf\Big\{C \ : \|c_\alpha(t)\| \le C\frac{t^\alpha e^{\omega t}}{\Gamma(\alpha+1)},\ t \ge 0\Big\} \le \|T\|,$$

whence we get the equality. ∎

5. Representations and the generation theorem for cosine functions

The main purpose in this section is to extend the Sova-Da Prato-Giusti theorem on generation of cosine functions (Theorem 5.3). Our generalization allows one to give a new proof of the classical generation theorem as a corollary of the variant of Kisyński theorem on $L^1_\omega(\mathbb{R}^+, *_c)$ (Theorem 3.1). Moreover, we may conjeture that the Sova-Da Prato-Giusti theorem and Theorem 3.1 are equivalent, see [3].

Let $\mathcal{A}$ be a Banach algebra and X a Banach space. A bounded algebra homomorphism from $\mathcal{A}$ into $\mathcal{B}(X)$ is called here a **representation**. Suppose that $\mathcal{A}$ is commutative and has a bounded approximate identity $\{e_n\}_{n\in\mathbb{N}}$. Given a representation $\Phi : \mathcal{A} \to \mathcal{B}(X)$, the **regularity space** $\mathcal{R}_\Phi$ is the closed linear span of $\{\Phi(a)x\colon a \in \mathcal{A},\ x \in X\}$, so that

$$\mathcal{R}_\Phi = \{x \in X\colon \lim_{n\to\infty} \Phi(e_n)x = x\}$$

(cf. [4]). By Cohen's theorem, $\mathcal{R}_\Phi = \{\Phi(a)x\colon a \in \mathcal{A},\ x \in X\}$ and there exists a unique representation $\hat{\Phi} : Mul(\mathcal{A}) \to \mathcal{B}(\mathcal{R}_\Phi)$, called the **extended representation** of Φ, such that

$$\hat{\Phi}(T_a)x = \Phi(a)x, \quad x \in \mathcal{R}_\Phi,$$

where $T_a(b) := ab$ for $a, b \in \mathcal{A}$. Moreover, $\|\Phi\| \le \|\hat{\Phi}\| \le (\liminf_n \|e_n\|)\,\|\Phi\|$, see [4, Theorem 2.4].

Consider now the case $\mathcal{A} = L^1_\omega(\mathbb{R}^+, *_c)$, where ω is an extendible weight function and $Mul(L^1_\omega(\mathbb{R}^+, *_c)) \cong M_\omega(\mathbb{R}^+, *_c)$ (Theorem 2.6). Then any representation $\Phi : L^1_\omega(\mathbb{R}^+, *_c) \to \mathcal{B}(X)$ can be extended to $\hat{\Phi} : M_\omega(\mathbb{R}^+, *_c) \to \mathcal{B}(\mathcal{R}_\Phi)$.

The set of Dirac measures $(\delta_t)_{t\ge0}$ plays an important role in relation to $L^1_\omega(\mathbb{R}^+, *)$: the family $(\delta_t)_{t\ge0}$ is contained in $M_\omega(\mathbb{R}^+, *)$; the semigroup property $\delta_t * \delta_s = \delta_{t+s}$ holds for $t, s \ge 0$ and $T_{\delta_t} f \to_{t\to0^+} f$ on $L^1_\omega(\mathbb{R}^+, *)$, where $T_{\delta_t} f(s) := \delta_t * f(s) = f(s-t)$ for $s, t \ge 0$. Here we discuss the role of Dirac measures $(\delta_t)_{t\ge0}$ in connection with $L^1_\omega(\mathbb{R}^+, *_c)$. We start by recalling some definitions and results about cosine functions which can be found in [1].

A map $C(\,\cdot\,) : [0,\infty) \to \mathcal{B}(X)$ satisfies the cosine functional equation if

$$2C(t)C(s) = C(t+s) + C(s-t), \quad s \ge t \ge 0,$$

and is a **cosine function** when, in addition, it is strongly continuous in $[0,\infty)$ and $C(0) = I_X$. An operator $(A, D(A))$ is the generator of a cosine function $(C(t))_{t\ge0}$, when

$$D(A) = \{x \in X\colon C(\,\cdot\,)x \in C^{(2)}([0,\infty), X)\}, \quad Ax = C''(0)x \ \text{ for } x \in D(A).$$

The generator of a cosine function $(A, D(A))$ is densely defined. Also, a cosine function is always exponentially bounded (i.e., there exist $M, \kappa \ge 0$ such that $\|C(t)\| \le Me^{\kappa t}$ for $t \ge 0$), and we have

$$\lambda(\lambda^2 - A)^{-1}x = \int_0^\infty e^{-\lambda t} C(t)x\,dt, \quad \lambda > \kappa,\ x \in X.$$

It is readily seen that $\delta_t \circ \delta_s = \delta_{s-t}$ if $s \ge t$; $\delta_t \circ \delta_s = 0$ if $s < t$ and

$$2(\delta_t *_c \delta_s) = \delta_{t+s} + \delta_{s-t}, \quad s \ge t \ge 0. \tag{5.1}$$

Lemma 5.1: *The family of operators* $(T_{\delta_t})_{t\geq 0}$ *defined by*

$$T_{\delta_t}(f) := \delta_t *_c f, \quad t \geq 0,\ f \in L^1_\omega(\mathbb{R}^+, *_c),$$

is a cosine function on $L^1_\omega(\mathbb{R}^+, *_c)$.

Proof. We write $C(t) := T_{\delta_t}$ and consider the map $C : [0,\infty) \to \mathcal{B}(L^1_\omega(\mathbb{R}^+, *_c))$. This map $C(\,\cdot\,)$ is well-defined (see Section 2) and the cosine equation holds from the equality (5.1). We need to check that the map $t \mapsto C(t)f$ is continuous on $L^1_\omega(\mathbb{R}^+, *_c)$. But this is a consequence of the equality

$$C(t)(f)(u) = \frac{1}{2}\left(\chi_{[t,\infty)}(u)f(u-t) + f(t+u) + \chi_{[0,t]}(u)f(t-u)\right),$$

for $t, u \geq 0$. ∎

In the next result we consider representations of $L^1_\omega(\mathbb{R}^+, *_c)$ on X; see [4, Theorem 3.3] for the case $L^1_\omega(\mathbb{R}^+, *)$.

Theorem 5.2: *For a representation* $\Phi : L^1_\omega(\mathbb{R}^+, *_c) \to \mathcal{B}(X)$, *let* $\mathcal{R}_\Phi$ *be the regularity space of* Φ *and let* $\hat{\Phi} : M_\omega(\mathbb{R}^+, *_c) \to \mathcal{B}(\mathcal{R}_\Phi)$ *be the extended representation of* Φ. *For each* $t \geq 0$, *put*

$$C(t) := \hat{\Phi}(T_{\delta_t}), \quad t \geq 0.$$

Then $(C(t))_{t\geq 0}$ *is a cosine function on* $\mathcal{R}_\Phi$ *such that* $\|C(t)\| \leq \|\hat{\Phi}\|\omega(t)$ *for* $t \geq 0$, *and*

$$\Phi(f)x = \int_0^\infty f(t)C(t)x dt,$$

for $x \in \mathcal{R}_\Phi$ *and* $f \in L^1_\omega(\mathbb{R}^+, *_c)$.

Proof. From Lemma 5.1, it is straightforward to check that $(C(t))_{t\geq 0}$ is a cosine function on $\mathcal{R}_\Phi$ and

$$\|C(t)\| \leq \|\hat{\Phi}\|\,\|\delta_t\|_\omega = \|\hat{\Phi}\|\omega(t), \quad t \geq 0.$$

Now we consider $\Phi : L^1_\omega(\mathbb{R}^+, *_c) \to \mathcal{B}(\mathcal{R}_\Phi)$ and define $\Psi : L^1_\omega(\mathbb{R}^+, *_c) \to \mathcal{B}(\mathcal{R}_\Phi)$ as

$$\Psi(f)x := \int_0^\infty f(t)C(t)x dt,$$

for $x \in \mathcal{R}_\Phi$ and $f \in L^1_\omega(\mathbb{R}^+, *_c)$. The map Ψ is a bounded algebra homomorphism ([13, Theorem 3.3]) and

$$\Psi(\epsilon_{-\lambda})x = \int_0^\infty e^{-\lambda t} C(t)x dt = \int_0^\infty e^{-\lambda t} \hat{\Phi}(T_{\delta_t})x dt = \hat{\Phi}\left(\int_0^\infty e^{-\lambda t} T_{\delta_t}\right)x$$
$$= \hat{\Phi}(T_{\epsilon_{-\lambda}})x = \Phi(\epsilon_{-\lambda})x,$$

for $x \in \mathcal{R}_\Phi$ and $\lambda > \kappa$, where κ is a bound of ω. Since the linear span of $(\epsilon_{-\lambda})_{\lambda>\kappa}$ is dense in $L^1_\omega(\mathbb{R}^+, *_c)$, we obtain that

$$\Phi(f)x = \int_0^\infty f(t)C(t)x dt,$$

for $x \in \mathcal{R}_\Phi$ and $f \in L^1_\omega(\mathbb{R}^+, *_c)$. ∎

Remark: In fact, it can be proved that

$$\hat{\Phi}(T_\mu)x = \int_0^\infty C(t)x d\mu(t), \quad x \in \mathcal{R}_\Phi,\ \mu \in M_\omega(\mathbb{R}^+, *_c),$$

along the same lines as in [4, Theorem 3.3].

Let ω be an extendible weight with bound $\kappa \geq 0$ and $r : (\kappa, \infty) \to \mathcal{B}(X)$ a function such that

$$r(\lambda)r(\mu) = \frac{1}{\lambda^2 - \mu^2}(\lambda r(\mu) - \mu r(\lambda)), \quad \lambda, \mu > \kappa. \tag{5.2}$$

We define the **regularity space** of r as

$$\mathcal{R}_r := \{x \in X : \lim_{\lambda\to\infty} \lambda r(\lambda)x = x\}.$$

It is clear that if $r : (\kappa, \infty) \to \mathcal{B}(X)$ satisfies (5.2), then $R : (\kappa^2, \infty) \to \mathcal{B}(X)$ given by

$$R(\lambda) := r(\sqrt{\lambda})/\sqrt{\lambda}, \quad \lambda > \kappa^2,$$

is a pseudo-resolvent on $\mathcal{B}(X)$. Moreover, $\mathcal{R}_R = \mathcal{R}_r$, where $\mathcal{R}_R$ is the regularity space of the pseudo-resolvent $(R(\lambda))_{\lambda>\kappa^2}$ defined by

$$\mathcal{R}_R := \{x \in X \ : \ \lim_{\lambda\to\infty} \lambda R(\lambda)x = x\}.$$

If $\|r\|_{W,\omega,\kappa} < \infty$, then there exists a representation $\phi : L^1_\omega(\mathbb{R}^+, *_c) \to \mathcal{B}(X)$ such that $r(\lambda) = \phi(\epsilon_{-\lambda})$ for $\lambda > \kappa$ (Theorem 3.1). It will be called **the representation associated with** r. Since $L^1_\omega(\mathbb{R}^+, *_c)$ has a bounded approximate identity (Proposition 1.4), we may consider the **regularity space** $\mathcal{R}_\phi$. It turns out that

$$\mathcal{R}_\phi = \mathcal{R}_r.$$

In particular, $\mathcal{R}_r$ is invariant for all $\phi(f)$ with $f \in L^1_\omega(\mathbb{R}^+)$, and any element of $\mathcal{R}_r$ can be represented as $\phi(f)x$ for some $f \in L^1_\omega(\mathbb{R}^+)$ and $x \in X$.

Now we give the announced generalization of the Sova-Da Prato-Giusti theorem.

Theorem 5.3 (Generalized Sova-Da Prato-Giusti theorem): *Let ω be an extendible weight with bound $\kappa \geq 0$, X a Banach space and $r : (\kappa, \infty) \to \mathcal{B}(X)$ a map such that*

$$r(\lambda)r(\mu) = \frac{1}{\lambda^2 - \mu^2}(\lambda r(\mu) - \mu r(\lambda)), \quad \lambda, \mu > \kappa,$$

in $\mathcal{B}(X)$ with

$$\|r\|_{W,\omega,\kappa} < \infty.$$

Let $\phi : L^1_\omega(\mathbb{R}^+, *_c) \to \mathcal{B}(X)$ be the representation of $L^1_\omega(\mathbb{R}^+, *_c)$ associated with r. Then there exists a unique cosine function $(C(t))_{t\geq 0}$ on $\mathcal{R}_r$ such that

$$C(t)\phi(f)x = \phi(T_{\delta_t}(f))x, \tag{5.3}$$

for $t \geq 0$, $f \in L^1_\omega(\mathbb{R}^+, *_c)$ and $x \in X$. Moreover, the cosine function $(C(t))_{t\geq 0}$ satisfies

$$r(\lambda)x = \int_0^\infty e^{-\lambda t} C(t)x dt,$$

for $\lambda > \kappa$, $x \in \mathcal{R}_r$ and $\|C(t)\| \leq M\omega(t)$ for $t \geq 0$.

Proof. Applying Theorem 5.2 to the representation $\phi : L^1_\omega(\mathbb{R}^+, *_c) \to \mathcal{B}(X)$ associated with r, we see that there exists a cosine function $(C(t))_{t\geq 0}$ on $\mathcal{R}_\phi (= \mathcal{R}_r)$ such that $\|C(t)\| \leq M\omega(t)$ for $t \geq 0$. The cosine function $(C(t))_{t\geq 0}$ satisfies

$$C(t)\phi(f) = \hat{\phi}(T_{\delta_t})\hat{\phi}(T_f) = \hat{\phi}(T_{\delta_t *_c f}) = \phi(T_{\delta_t}(f)),$$

for $t \geq 0$ and $f \in L^1_\omega(\mathbb{R}^+, *_c)$ and

$$r(\lambda)x = \phi(\epsilon_{-\lambda})x = \int_0^\infty e^{-\lambda t} C(t)x dt,$$

for $x \in \mathcal{R}_r$ and $\lambda > \kappa$. Finally, the uniqueness of $(C(t))_{t\geq 0}$ follows from (5.3) and the fact that every element of $\mathcal{R}_r$ can be represented as $\phi(f)x$ for some $f \in L^1_\omega(\mathbb{R}^+, *_c)$ and $x \in X$. ∎

Let $R = (R(\lambda))_{\lambda>\kappa}$ be a pseudo-resolvent on $\mathcal{B}(X)$. It is known that the kernel and range of $R(\lambda)$ are independent of λ, so we denote them by $\ker(R)$ and $\mathrm{Im}(R)$ respectively. Note that $(R(\lambda))_{\lambda>\kappa}$ is the resolvent of a densely defined closed operator $(A, D(A))$, i.e. $R(\lambda) = (\lambda - A)^{-1}$, if and only if $\ker(R) = \{0\}$ and $\overline{\mathrm{Im}(R)} = X$.

We conclude the paper by deducing the Sova-Da Prato-Giusti theorem from Theorem 5.3.

THEOREM 5.4: *Let $(A, D(A))$ be a densely defined operator on a Banach space X. The following assertions are equivalent:*

(i) *$(A, D(A))$ generates a cosine function.*

(ii) *There exist $\kappa, M \geq 0$ such that $(\kappa^2, \infty) \subset \rho(A)$ and*

$$\frac{1}{n!}\left|(\lambda-\kappa)^{n+1}\frac{d^n}{d\lambda^n}(\lambda(\lambda^2 - A)^{-1})\right| \leq M,$$

for all $\lambda > \kappa$ and $n \in \mathbb{N} \cup \{0\}$.

Proof. (i) ⇒ (ii) There exist $\kappa \geq 0$ and $M > 0$ such that $\|C(t)\| \leq Me^{\kappa t}$. Take $\omega(t) := e^{\kappa t}$ for $t \geq 0$ and define $\Phi : L^1_\omega(\mathbb{R}^+, *_c) \to \mathcal{B}(X)$ by

$$\Phi(f)x := \int_0^\infty f(t)C(t)x dt,$$

for $x \in X$ and $f \in L^1_\omega(\mathbb{R}^+, *_c)$. Note that

$$\Phi(\epsilon_{-\lambda}) = \lambda(\lambda^2 - A)^{-1}, \quad \lambda > \kappa,$$

and $\|t^n e^{-\lambda t}\|_\omega = n!/(\lambda - \kappa)^{n+1}$ for $n \in \mathbb{N} \cup \{0\}$ and $\lambda > \kappa$. Applying Theorem 0.1, we obtain

$$\frac{1}{n!}\left|(\lambda-\kappa)^{n+1}\frac{d^n}{d\lambda^n}(\lambda(\lambda^2 - A)^{-1})\right| \leq M',$$

for all $\lambda > \kappa$ and $n \in \mathbb{N} \cup \{0\}$.

(ii) ⇒ (i) Let $r(\lambda) := \lambda(\lambda^2 - A)^{-1}$ for $\lambda > \kappa$. Then

$$r(\lambda)r(\mu) = \frac{1}{\lambda^2 - \mu^2}(\lambda r(\mu) - \mu r(\lambda)), \quad \lambda, \mu > \kappa,$$

$\|r\|_{W,\omega,\kappa} < \infty$ with $\omega(t) = e^{\kappa t}$ for $t \geq 0$. Then, applying Theorem 5.3, we obtain a cosine family $(C(t))_{t\geq 0}$ on $\mathcal{R}_r$. Note that $\mathcal{R}_r = \overline{\mathrm{Im}(R)} = X$ (where $R(\mu) = (\mu - A)^{-1}$ for $\mu > \kappa^2$). It is easy to check that $(A, D(A))$ is the generator of $(C(t))_{t\geq 0}$. ∎

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